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Static Perfect Fluid in Brans–Dicke Theory

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The static perfect fluid in Brans–Dicke theory with spherical symmetry and conformal flatness leads to a differential equation in terms of the scalar field only. We obtain a unique exact solution for the case $p=\varepsilon\rho$, but density and pressure are singular at the center. We further consider the metric corresponding to a static nonrotating space-time with two mutually orthogonal spacelike Killing vectors in Brans–Dicke theory. We obtain a differential equation involving only the scalar field for the equation of state $p=\varepsilon\rho$. The general solution is found as a transcendental function. Finally, we generalize a theorem given by Bronnikov and Kovalchuk (1979) for perfect fluid in Einstein's theory.

1. INTRODUCTION

Static perfect fluid distributions with high symmetries such as spherical, cylindrical, and planar symmetries are widely discussed in the literature (Schwarzschild, 1916; Tolman, 1939; Marder, 1958; Misner and Zapolsky, 1964; Evans, 1977; Bronnikov and Kovalchuk, 1979; and references therein). However, the corresponding problem in Brans–Dicke theory still lacks thorough investigations, although it deserves attention because Brans–Dicke theory is perhaps the best motivated scalar tensor theory of gravitation in spite of some recent experiments giving evidences against this theory.

In Sections 2, 3, and 4 we discuss spherically symmetric solutions. We arrive at a complicated nonlinear differential equation involving the only scalar field variable for the most general conformally flat perfect fluid distribution in Brans–Dicke theory. Only in a special case of an equation of state $p=\varepsilon\rho$ we could give the exact solution, which is unique but singular in the sense that the density and pressure are infinitely large at the center. This

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solution is exactly one of the special cases of the singular solution given previously by Bruckman and Kazes (1979) showing that their solution includes the conformally flat solution.

In Sections 5 and 6 we consider a metric corresponding to a static nonrotating space-time with two mutually orthogonal spacelike Killing vectors. This metric may be interpreted to represent cylindrical, toroidal, planar, or pseudoplanar symmetry depending on the behavior of the coordinates. We obtain the most general nonlinear differential equation involving only the scalar field variable for a perfect fluid with the isothermal equation of state $p=\varepsilon\rho$, ε being a constant. The general solution was obtained as a transcendental function and explicit solutions can only be obtained in certain special cases. It has also been proved that there cannot exist a perfect fluid distribution satisfying an equation of state $p=\varepsilon\rho$ and also having planar or pseudoplanar symmetry along with additional mirror symmetry. This may be said to be the generalization of the corresponding theorem of Bronnikov and Kovalchuk (1979) for a perfect fluid in Einstein's theory.

Finally in Section 7 we present the explicit solutions of Einstein–Brans–Dicke equations in empty space for the line element given in section 5. The static vacuum solutions for spherical symmetry were already given by Brans (1962).

2. CONFORMALLY FLAT STATIC SPHERICALLY SYMMETRIC METRIC

The general static spherically symmetric metric may be written as

$$ds^2=e^{2\sigma(r)}\left[-V^2(r)dt^2+dr^2+r^2\,d\Omega^2\right] \tag{1}$$

In order that a four-dimensional space-time be conformally flat all the components of the associated Weyl tensor (Eisenhart, 1966),

$$C^{\alpha}_{\delta\gamma\beta}=R^{\alpha}_{\delta\gamma\beta}+\tfrac{1}{2}\left(\delta^{\alpha}_{\gamma}R_{\delta\beta}-\delta^{\alpha}_{\beta}R_{\delta\gamma}+g_{\delta\beta}R^{\alpha}_{\gamma}-g_{\delta\gamma}R^{\alpha}_{\beta}\right)+(R/6)\left(\delta^{\alpha}_{\beta}g_{\delta\gamma}-\delta^{\alpha}_{\gamma}g_{\delta\beta}\right) \tag{2}$$

must vanish. If this condition is satisfied for

$$ds^2=-V^2(r)dt^2+dr^2+r^2\,d\Omega^2 \tag{3}$$

then (1) is consequently conformally flat. Greek indices range from 0, 1, 2, 3 and latin indices 1, 2, 3.

Substituting the metric given by (3) into (2), we obtain the components of the Weyl tensor which do not vanish identically,

$$C_{hook} = -\tfrac{1}{2} V V_{;hk} + \tfrac{1}{6} g_{hk} V g^{ab} V_{;ab} = 0 \tag{4}$$

$$C_{hijk} = \frac{1}{2V}\left[g_{hj} V_{;ik} - g_{hk} V_{;ij} + g_{ik} V_{;hj} - g_{ij} V_{;hk}\right] + \frac{g^{ab} V_{;ab}}{3V}\left(g_{hk} g_{ij} - g_{hj} g_{ik}\right) = 0 \tag{5}$$

Substituting equation (4) into (5) we see that the components (5) are identically zero; then we are left only with equation (4), which may be split into

$$\tfrac{1}{3} g^{ab} V_{;ab} = V_{;hk}; \qquad \text{for } h = k$$

$$V_{;hk} = 0, \qquad \text{for } h = k \tag{6}$$

The system of equations (6) is reducible to

$$\frac{1}{3}\left(V'' + \frac{2V'}{r}\right) = V'' = \frac{V'}{r} \tag{7}$$

where the prime means differentiation with respect to r. The general solution of (7) is

$$V = ar^2 + b \tag{8}$$

where a and b are constants of integration. Now we can write the general spherically symmetric conformally flat metric as

$$ds^2 = e^{2\sigma(r)}\left[-(ar^2 + b)^2\, dt^2 + dr^2 + r^2\, d\Omega^2\right] \tag{9}$$

3. PERFECT FLUID AND BRANS–DICKE THEORY IN A SPHERICALLY SYMMETRIC CONFORMALLY FLAT SPACE-TIME

Brans–Dicke field equations with perfect fluid are

$$G_\alpha^\beta = -\frac{k}{\phi} T_\alpha^{\ \beta} - \frac{\omega}{\phi^2}\left(\phi_{,\alpha}\phi^{,\beta} - \frac{1}{2}\delta_\alpha^{\ \beta}\phi_{,\gamma}\phi^{,\gamma}\right) - \frac{1}{\phi}\left(\phi_{;\alpha}^{;\beta} - \delta_\alpha^\beta \Box\phi\right) \tag{10}$$

$$\Box\phi = \frac{k}{3 + 2\omega} T_\alpha^\alpha \tag{11}$$

ϕ is the scalar field and $T_{\alpha\beta}$ is the energy–momentum tensor for a perfect fluid, so that

$$T_{\alpha\beta}=(\rho+p)u_\alpha u_\beta+pg_{\alpha\beta} \tag{12}$$

where ρ is the mass density, p is the pressure, and u_α is the 4-velocity satisfying

$$u_\alpha u^\alpha=-1 \tag{13}$$

For the static case $u_i=0$.

By considering the space-time (9), the field equations (10) become

$$G_1^1=-e^{-2\sigma}\left[3\sigma'^2+\frac{4ar\sigma'}{b+ar^2}+\frac{4\sigma'}{r}+\frac{4a}{b+ar^2}\right]$$

$$=-\frac{k}{\phi}p+e^{-2\sigma}\left[-\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2-\frac{\phi''}{\phi}+\sigma'\frac{\phi'}{\phi}\right]+\frac{\Box\phi}{\phi} \tag{14}$$

$$G_2^2=G_3^3=-e^{-2\sigma}\left[2\sigma''+\sigma'^2+\frac{2\sigma'}{r}+\frac{4ar\sigma'}{b+ar^2}+\frac{4a}{b+ar^2}\right]$$

$$=-\frac{k}{\phi}p+e^{-2\sigma}\left[\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2-\left(\frac{1}{r}+\sigma'\right)\frac{\phi'}{\phi}\right]+\frac{\Box\phi}{\phi} \tag{15}$$

$$G_4^4=-e^{-2\sigma}\left[2\sigma''+\sigma'^2+\frac{4\sigma'}{r}\right]$$

$$=\frac{k}{\phi}\rho+e^{-2\sigma}\left[\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2-\left(\frac{2ar}{b+ar^2}+\sigma'\right)\frac{\phi'}{\phi}\right]+\frac{\Box\phi}{\phi} \tag{16}$$

and (11),

$$\Box\phi=e^{-2\sigma}\left[\phi''+\phi'\left(2\sigma'+\frac{2}{r}+\frac{2ar}{b+ar^2}\right)\right]$$

$$=\frac{k}{3+2\omega}(3p-\rho) \tag{17}$$

A further equation that we can build which may be helpful, however not independent, is the trace of (10) which gives

$$G^{\alpha}_{\alpha} = -e^{-2\sigma}\left[6\sigma'' + 6\sigma'^2 + \frac{12\sigma'}{r} + \frac{12a(r\sigma'+1)}{b+ar^2}\right]$$

$$= -\frac{k}{\phi}T^{\alpha}_{\alpha} + e^{-2\sigma}\omega\left(\frac{\phi'}{\phi}\right)^2 + 3\frac{\Box\phi}{\phi} \tag{18}$$

The above system of equations has four unknowns, σ, ϕ, ρ, and p, and four independent equations, (14)–(17), which means that in principle it may be soluble.

Subtracting equation (14) from (15) we obtain

$$-\sigma'' + \sigma'^2 + \frac{\sigma'}{r} - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 + \frac{1}{2r}\frac{\phi'}{\phi} + \sigma'\frac{\phi'}{\phi} - \frac{1}{2}\frac{\phi''}{\phi} = 0 \tag{19}$$

Substituting (17) into (18), instead of (16), we have

$$3\sigma'' + 3\sigma'^2 + \frac{6\sigma'}{r} + \frac{6a(r\sigma'+1)}{b+ar^2}$$

$$+\omega\left[\frac{1}{2}\left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} - \frac{\phi'}{\phi}\left(2\sigma' + \frac{2}{r} + \frac{2ar}{b+ar^2}\right)\right] = 0 \tag{20}$$

Now considering the transformations

$$\sigma' = \alpha \tag{21}$$

$$\frac{\phi'}{\phi} = \psi \tag{22}$$

and applying them in (19) and (20) we obtain

$$\alpha' = \alpha^2 + \alpha\left(\frac{1}{r} + \psi\right) - \frac{1}{2}\psi' + \frac{1}{2r}\psi - \frac{\omega+1}{2}\psi^2 \tag{23}$$

and

$$\alpha' = -\alpha^2 + \alpha\left(-\frac{2}{r} - \frac{2ar}{b+ar^2} + \frac{2}{3}\omega\psi\right) + \frac{\omega}{3}\psi'$$

$$+\psi\left(\frac{2}{3}\frac{\omega}{r} + \frac{2}{3}\frac{\omega ar}{b+ar^2}\right) + \frac{\omega}{6}\psi^2 - \frac{2a}{b+ar^2} \tag{24}$$

We can build one differential equation only in terms of ψ from the two Riccati equations for α, (23) and (24). Thus the general spherically symmetric conformally flat solution for a perfect fluid in Brans–Dicke theory is uniquely given by (23) and (24), and in the following section we will consider a special class of such solutions.

4. SOLUTION WITH AN EQUATION OF STATE

One particular solution of the system (23), (24) may be found by assuming the equation of state

$$p=\varepsilon\rho \tag{25}$$

where ε is a positive constant. It has been proved (Banerjee and Bhattacharya, 1979) that in this case

$$\phi=A(g_{00})^{c/2}=Ae^{c\sigma}(ar^2+b)^c \tag{26}$$

where A and c are constants. Substituting (26) into (23) and (24) by taking into consideration the transformations (21) and (22) we obtain

$$\left(\frac{1}{c}+\frac{1}{2}\right)\psi'=\left(\frac{1}{c^2}+\frac{1}{c}-\frac{\omega+1}{2}\right)\psi^2+\left(2f+\frac{1}{r}\right)\left(\frac{1}{c}+\frac{1}{2}\right)\psi \tag{27}$$

$$\left(\frac{1}{c}-\frac{\omega}{3}\right)\psi'=\left(-\frac{1}{c^2}+\frac{2\omega}{3c}+\frac{\omega}{6}\right)\psi^2-\left(f+\frac{2}{r}\right)\left(\frac{1}{c}-\frac{\omega}{3}\right)\psi-f'-\frac{f}{r} \tag{28}$$

where

$$f=-\frac{2ar}{ar^2+b} \tag{29}$$

Considering first solutions for $a=0$, which implies $f=0$, then ψ that satisfies (27) and (28) is

$$\psi=N\frac{1}{r} \tag{30}$$

where

$$N=\frac{1/c-\omega/3}{-1/c^2+2\omega/3c+\omega/6} \tag{31}$$

and the relation between c and ω is

$$\frac{1}{c^3} - \frac{\omega}{c^2} + \left(-\frac{\omega}{6} + \frac{1}{2}\right)\frac{1}{c} - \frac{\omega}{3} - \frac{\omega^2}{6} = 0 \tag{32}$$

Since

$$c = \frac{3\varepsilon - 1}{(2\omega + 3) + (\omega + 1)(3\varepsilon - 1)}$$

(Banerjee and Bhattacharya, 1979) the equation (32) actually determines a relationship between ε and ω. The second situation is for $a \neq 0$, which implies $f \neq 0$; then the solution of equation (27) is

$$\psi = -Mf \tag{33}$$

where

$$M = \frac{1/c + 1/2}{1/c^2 + 1/c - (\omega + 1)/2} \tag{34}$$

Now substituting (33) into (28) we obtain the relation

$$f = -\frac{B}{r} \tag{35}$$

where B is a constant given by

$$B = \frac{3M - 2/(1/c - \omega/3)}{2M - 1/(1/c - \omega/3) + M^2/N} \tag{36}$$

and where N is given by (31).

Comparing equation (35) with the definition of f given in (29) we see that (35) can only be true if $b = 0$. It is possible to show that even in this case the line element (9) for $a \neq 0$ and $b = 0$ may, by a coordinate transformation like $\bar{r} = 1/ar$, be written in the form $ds^2 = e^{2\sigma(\bar{r})}[-dt^2 + d\bar{r}^2 + \bar{r}^2\, d\Omega^2]$. Hence we conclude that both situations, $f = 0$ and $f \neq 0$, are equivalent. We can write without loss of generality for our problem the line element (9) as

$$ds^2 = e^{2\sigma(r)}[-dt^2 + dr^2 + r^2\, d\Omega^2] \tag{37}$$

Using equations (26) and (30) we have the solutions for the conformal factor and the scalar field

$$e^{2\sigma}=c_1^2r^{2N/c} \tag{38}$$

$$\phi=c_2r^N \tag{39}$$

where c_1 and c_2 are integration constants.

So finally we have arrived at the result that the metric (38) along with (39) is the only conformally flat solution in Brans–Dicke theory for a perfect fluid which is consistent with an equation of state $p=\varepsilon\rho$. However the solution is singular, the mass density and pressure attain an infinitely large value at the center of the sphere located at $\bar{r}=re^{\sigma}=0$.

At this end it is interesting to note that our solution (38) given in isotropic coordinates can be transformed in the usual curvature coordinate by a simple coordinate transformation

$$\bar{r}=re^{\sigma}=c_1r^{(1+N/c)} \tag{40}$$

and the line element can be written in the form

$$ds^2=-(c_1)^{2/(1+N/c)}(\bar{r})^{2N/c/(1+N/c)}dt^2+\frac{1}{(1+N/c)^2}d\bar{r}^2+\bar{r}^2d\Omega^2 \tag{41}$$

This is exactly of the same form as given in the paper of Bruckman and Kazes (1977). In fact the solution given in their paper is conformally flat if the power of $\bar{r}$ in g_{00} is related to g_{11}, which is a constant quantity, exactly in the same way as in (41), and this in turn implies a specific relationship between ε and ω.

5. STATIC CYLINDRICAL AND PLANE SYMMETRIC SOLUTIONS

We consider a static space-time possessing two spacelike Killing vectors which are mutually orthogonal and also orthogonal to the timelike Killing vector. The metric can then be chosen as

$$ds^2=-e^{2\gamma(x)}dt^2+e^{2\lambda(x)}dx^2+e^{2\mu(x)}d\eta^2+e^{2\beta(x)}d\xi^2 \tag{42}$$

The above metric corresponds to cylindrical symmetry if ξ and η represent the azimuthal and longitudinal coordinates, respectively, so that $\xi\in(0,2\pi)$ and $\eta\in(-\infty,\infty)$. If both ξ and η are angular coordinates, it represents toroidal symmetry, whereas if both ξ and η represent longitudinal coordinates [$\xi \in (-\infty, +\infty)$; $\eta \in (-\infty, +\infty)$], the metric (42) represents pseudo-planar symmetry or planar symmetry. In the following we attempt to find exact solutions of the field equations in Brans–Dicke theory corresponding to the metric (42) for a perfect fluid distribution. One can, however, use without the loss of generality the coordinate condition

$$\lambda = \gamma + \mu + \beta \tag{43}$$

The above coordinate condition enables us to write the field equations in a symmetric form.

The field equations are

$$G_1^1 = -e^{-2\lambda} U = -\frac{kp}{\phi} + e^{-2\lambda}\left[-\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 + \lambda'\frac{\phi'}{\phi}\right] \tag{44}$$

$$G_2^2 = -e^{-2\lambda}[\beta'' + \gamma'' - U] = -\frac{kp}{\phi} + e^{-2\lambda}\left[\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 - \mu'\frac{\phi'}{\phi} + \frac{\phi''}{\phi}\right] \tag{45}$$

$$G_3^3 = -e^{-2\lambda}[\gamma'' + \mu'' - U] = -\frac{kp}{\phi} + e^{-2\lambda}\left[\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 - \beta'\frac{\phi'}{\phi} + \frac{\phi''}{\phi}\right] \tag{46}$$

$$G_4^4 = -e^{-2\lambda}[\beta'' + \mu'' - U] = \frac{k\rho}{\phi} + e^{-2\lambda}\left[\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 - \gamma'\frac{\phi'}{\phi} + \frac{\phi''}{\phi}\right] \tag{47}$$

where the primes indicate differentiation with respect to x and

$$U = \beta'\gamma' + \beta'\mu' + \gamma'\mu' \tag{48}$$

The field equation (11) becomes

$$\Box\phi = e^{-2\lambda}\phi'' = \frac{k}{3+2\omega}(-\rho + 3p) \tag{49}$$

Here we assume again that the equation of state for the fluid is

$$p = \varepsilon\rho \tag{50}$$

Now we attempt to solve this system of five equations and five unknown functions γ, μ, β, ρ, and ϕ. Subtracting equation (45) from (46) and integrating, we obtain

$$(\mu - \beta)' = \frac{D_1}{\phi} \tag{51}$$

where D_1 is a constant of integration. Subtracting (46) from (47) and applying (48) and (49) we obtain

$$(\beta-\gamma)''=h\frac{\phi''}{\phi}-(\beta-\gamma)'\frac{\phi'}{\phi} \tag{52}$$

where

$$h=-\frac{(1+\varepsilon)(3+2\omega)}{3\varepsilon-1} \tag{53}$$

Adding up (44) and (45) and applying (49) and (50) we have

$$(\beta+\gamma)''=j\frac{\phi''}{\phi}-(\beta+\gamma)''\frac{\phi'}{\phi} \tag{54}$$

where

$$j=\frac{2\varepsilon(3+2\omega)}{3\varepsilon-1}-1 \tag{55}$$

Multiplying (52) by j and subtracting from (54) multiplied by h and integrating we have

$$h(\beta+\gamma)'-j(\beta-\gamma)'=\frac{D_2}{\phi} \tag{56}$$

where D_2 is a constant of integration.

Equating (51) with (56) and integrating we finally have

$$\beta[D_1(h-j)+D_2]+\gamma D_1(h+j)-\mu D_2-D_3=0 \tag{57}$$

where D_3 is a constant of integration.

Then from (57) we prove that the exponentials β, γ, and μ must be linearly connected for a static perfect fluid having equation of state (50) in Brans–Dicke theory.

In view of the Bianchi identity we have $T^{\mu\nu}_{;\nu}=0$, which in turn yields

$$p'=-(\rho+p)\gamma' \tag{58}$$

Applying the equation of state (50) into (58) and integrating we have

$$\exp\left(-\frac{1+\varepsilon}{\varepsilon}\gamma\right)=D_4\rho \tag{59}$$

where D_4 is a constant of integration.

Here is valid too the relation (26) which allows us to write

$$\left(\frac{\phi}{A}\right)^{1/c}=e^{\gamma} \tag{60}$$

Equations (59) and (60) give

$$\left(\frac{\phi}{A}\right)^{-(1/c)[(1+\varepsilon)/\varepsilon]}=D_4\rho \tag{61}$$

Substituting (61) for ρ into (49) and applying (50) we have

$$e^{2\lambda}=\phi''\frac{3+2\omega}{k(3\varepsilon-1)}D_4\left(\frac{\phi}{A}\right)^{(1/c)[(1+\varepsilon)/\varepsilon]} \tag{62}$$

Differentiating (62) with respect to x and using (43), (51), and (56) in order to obtain the equation written in terms of γ and ϕ and then using (60), we finally have

$$\phi\phi'''+m\phi''\phi'+n\phi''=0 \tag{63}$$

where

$$m=\frac{1}{c}\left[\frac{1-\varepsilon}{\varepsilon}-\frac{2(h+j)}{h-j}\right] \tag{64}$$

$$n=-\frac{2D_2}{h-j}-D_1 \tag{65}$$

We can integrate once (63), giving

$$\phi\phi''+\frac{m-1}{2}\phi'^2+n\phi'+D_5=0 \tag{66}$$

where D_5 is a constant of integration. We observe that if we have $\mu(x)=\beta(x)$ and apply (60) into (66), the differential equation that we obtain has the same form as that obtained for g_{00} by Banerjee and Bhattacharya (1979).

Now considering the transformation

$$\phi\phi'=\frac{1}{y(\phi)} \tag{67}$$

applied to (66) gives us

$$\frac{dy}{d\phi}+\left(1-\frac{m-1}{2}\right)\frac{y}{\phi}-ny^2-D_5\phi y^3=0 \tag{68}$$

If $n \neq 0$ we can make another transformation

$$y(\phi)\phi^{1-(m-1)/2} = \Omega(\alpha) \tag{69}$$

$$\alpha = n\frac{\phi^{(m-1)/2}}{(m-1)/2} \tag{70}$$

then (68) becomes

$$\frac{d\Omega}{d\alpha} = \frac{D_5}{n^2}\left(\frac{m-1}{2}\right)\alpha\Omega^3 + \Omega^2 \tag{71}$$

Lastly, assuming

$$\frac{d\alpha}{dw} = -\frac{1}{w\Omega(\alpha)} \tag{72}$$

leads us to have (71) transformed into the differential equation

$$w^2\frac{d^2\alpha}{w^2} + \frac{D_5}{n^2}\left(\frac{m-1}{2}\right)\alpha = 0 \tag{73}$$

which has the general solution,

$$\frac{\alpha}{\sqrt{w}} = \begin{cases} B_1\cos(q\ln w) + B_2\sin(q\ln w), & \text{for } q^2 = s - \frac{1}{4} > 0 \\ B_1 w^q + B_2 w^{-q}, & \text{for } q^2 = \frac{1}{4} - s > 0 \\ B_1 + B_2 \ln w, & \text{for } s = \frac{1}{4} \end{cases} \tag{74}$$

where

$$s = \frac{D_5}{n^2}\frac{m-1}{2}$$

and B_1 and B_2 are constants of integration.

If $n = 0$ then the solution of (68) is

$$y = \begin{cases} \left[B_1\phi^{3-m} - \dfrac{D_5}{(m-1)/2}\phi^2\right]^{-1/2}, & \text{for } m \neq 1 \\ \left[B_1\phi^2 - 2D_5\phi^2\ln\phi\right]^{-1/2}, & \text{for } m = 1 \end{cases} \tag{75}$$

In general it is not possible to go back from solutions (74) and (75) by applying the inverse transformations to obtain $\phi(x)$ because of the appearance of transcendental equations but, nonetheless, it is possible to obtain particular solutions of (66) by choosing particular values for the constants which appear in the general solutions (74) and (75).

6. A LEMMA AND A THEOREM FOR PLANAR SYMMETRY

In this section we propose to state a theorem regarding plane symmetric solutions for a perfect fluid with an isothermal equation of state in Brans–Dicke theory. It may be said to be a generalization of the corresponding theorem proved by Bronnikov and Kovalchuk (1979) in Einstein's theory.

We know that pseudoplanar symmetry plus mirror symmetry demand that

$$p' = \gamma' = \mu' = \beta' = 0 \tag{76}$$

at some plane $x=0$ without the loss of generality. In view of equations (51) and (56), condition (76) leads us to

$$D_1 = D_2 = 0 \tag{77}$$

Since $D_1=0$, we have $\mu'=\beta'$ or $\mu=\beta+\text{const}$, and by a suitable scale transformation of a coordinate, μ can be made equal to β, which gives planar symmetry. This leads us to the following lemma:

> *Lemma.* Static systems with pseudoplanar plus mirror symmetry such that $(T_2^2 = T_3^3)_{\text{matter}}$ are necessarily planarly symmetric in Brans–Dicke theory.

This is a generalization of the corresponding result in Einstein's theory (Bronnikov and Kovalchuk, 1979).

We observe that this lemma has been proved whenever $(T_2^2)_{\text{matter}} = (T_3^3)_{\text{matter}}$ and is without any interference of the equation of state (50).

Further, in view of (43) one has

$$\lambda' = \gamma' + \mu' + \beta' \tag{78}$$

and with the help of (51) and (56), (78) becomes

$$\lambda' = \gamma'\left(1 - 2\frac{h+j}{h-j}\right) + \left(D_1 + \frac{2D_2}{h-j}\right)\frac{1}{\phi} \tag{79}$$

Differentiating (60) with respect to x and introducing it into (79) we finally have

$$\lambda' = \frac{1}{c}\frac{\phi'}{\phi}\left[1 - 2\left(\frac{h+j}{h-j}\right)\right] + \left(D_1 + \frac{2D_2}{h-j}\right)\frac{1}{\phi} \tag{80}$$

The condition (76) applied to (78) implies that $\lambda' = 0$ at $x = 0$; then with (77) and (80) we have $\phi' = 0$ at $x = 0$ because $D_1 = D_2 = 0$ in view of mirror symmetry. Considering the equation of state $p = \varepsilon\rho$ and (44) with the conditions obtained above we obtain that $\rho = 0$ at $x = 0$. Further, equation (49) leads us also to $\phi'' = 0$ at $x = 0$ and then by (66) it implies that at $x = 0$, $D_5 = 0$. Hence equation (66) reduces to

$$\phi\phi'' + \frac{m-1}{2}\phi'^2 = 0 \tag{81}$$

Equation (81) after integration may be written

$$\phi = (E_1 x + E_2)^{2/(m+1)} \tag{82}$$

where E_1 and E_2 are constants of integration. The condition $\phi' = 0$ at $x = 0$ imposes $E_1 = 0$ in (82), which gives a flat solution by (60). This leads us to the following theorem:

> *Theorem.* There cannot exist a static, planarly, or pseudoplanarly symmetric perfect fluid distribution with the equation of state $p = \varepsilon\rho$ with additional mirror symmetry in Brans–Dicke theory.

This is a generalization of the corresponding theorem for perfect fluid in Einstein's theory of gravitation (Bronnikov and Kovalchuk, 1979).

7. VACUUM BRANS–DICKE SOLUTIONS FOR CYLINDRICAL AND PLANAR SYMMETRY

In the absence of any matter one should put $p = \rho = 0$ in the field equations (44)–(47). The wave equation for the scalar field now reduces to $\Box\phi = 0$, which in turn yields the solution for the scalar field ϕ after a scale transformation of x coordinate in the form

$$\phi = x + c \tag{83}$$

The field equations (44)–(47) after putting $p = \rho = 0$ finally yield the following relations:

$$\beta''-\mu''=-(\beta'-\mu')\frac{\phi'}{\phi} \tag{84}$$

$$\gamma''-\beta''=-(\gamma'-\beta')\frac{\phi'}{\phi} \tag{85}$$

$$-\beta''-\gamma''=(\beta'+\gamma')\frac{\phi'}{\phi} \tag{86}$$

Adding equations (85) and (86) and integrating, we obtain

$$\beta'=\frac{D_1}{\phi} \tag{87}$$

Now introducing (87) into (85) and (84) and integrating we have

$$\gamma'=\frac{D_2}{\phi} \qquad \text{and} \qquad \mu'=\frac{D_3}{\phi} \tag{88}$$

where D_1, D_2, D_3 are all constants and ϕ is given by (83). The equations (87) and (88) yield the complete solutions on simple integrations and suitable scale transformations of the coordinates x, ξ and η as

$$\begin{aligned} e^{\beta}&=(x+c)^{D_1}\\ e^{\gamma}&=(x+c)^{D_2}\\ e^{\mu}&=(x+c)^{D_3} \end{aligned} \tag{89}$$

The solutions (89) satisfy all the Brans–Dicke field equations provided

$$(D_1D_2+D_2D_3+D_3D_1)=\omega/2-(D_1+D_2+D_3)$$

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